Quantum mechanics

Q system: Described by N
"class. possibilities".

Q state: Vector $|\psi\rangle$ in $\mathbb{C}^N$
with $\| |\psi\rangle \| = 1$

Intuition: $\psi_i \hat{=}$ amplitude of poss i
$|\psi_i^2| = |\psi_i|^2 \hat{=}$ prob. of poss i.

Operations on Q states

- Should be function
$U : \mathbb{C}^N \longrightarrow \mathbb{C}^N$
- Linear ($U(\psi + \phi) = U\psi + U\phi$,
$U(\alpha\psi) = \alpha U\psi$)
- Maps q. states to q. states
$\| |\psi\rangle \| = 1 \Rightarrow \| U|\psi\rangle \| = 1$

Equiv to: U is length-pres.
"isometry"

We consider: linear isometries $\mathbb{C}^N \to \mathbb{C}^N$

Lin. func. is isometry iff

$$\underbrace{U^+}_{\text{conjugate transpose}} U = 1$$

Lin. func. is unitary iff

$$U^+U = 1 \wedge UU^+ = 1$$

For final dim. spaces and if dom + codomain same, then "isometry $\Leftrightarrow$ unitary"

Def An op. on q state is simply a linear unitary op. on $\mathbb{C}^N$

Examples: $X: |0\rangle \to |1\rangle$

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix} \qquad |1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

$X: |1\rangle \to |0\rangle$

$$X = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$X^+X = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \cdot \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = 1$$

"Set to 1" $U|0\rangle \to |1\rangle$

$U|1\rangle \to |1\rangle$

$$U = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$$

Not unitary!

$$U\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$U\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$Ua = Ub$, $a \neq b$

$$\Rightarrow U(\underbrace{a-b}_{\neq 0}) = 0$$

Every unitary is injective,

"reversible"

$$Z = \begin{pmatrix} 1 & \\ & -1 \end{pmatrix} \qquad Z|0\rangle = |0\rangle$$
$$Z|1\rangle = -|1\rangle$$

$$Y = \begin{pmatrix} & -i \\ i & \end{pmatrix} \qquad \sigma_x \; \sigma_y \; \sigma_z$$

Complete meas. in computational basis

Given $|\psi\rangle$ on a system with poss. $1, \ldots, N$

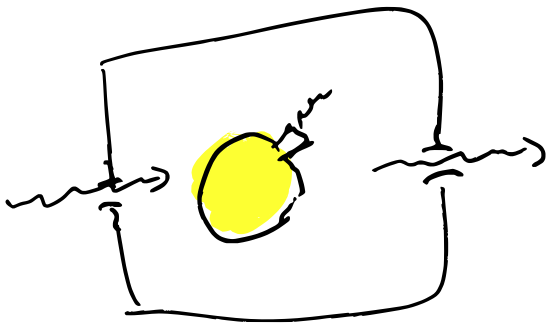
We ask: "which class. poss. is the system in?"

Called <u>measurement</u>

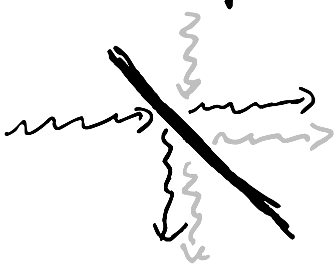
$$\Pr[\text{outcome } i] = |\psi_i|^2$$
$(i = 1, \ldots, N)$

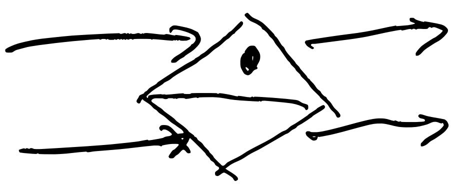
State after meas. outcome i (post-meas-state): $|i\rangle = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow$ i-th pos.

Elitzur-Vaidman Bomb Tester



Main component: Beam splitter

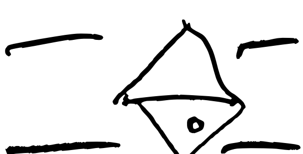




$$B = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

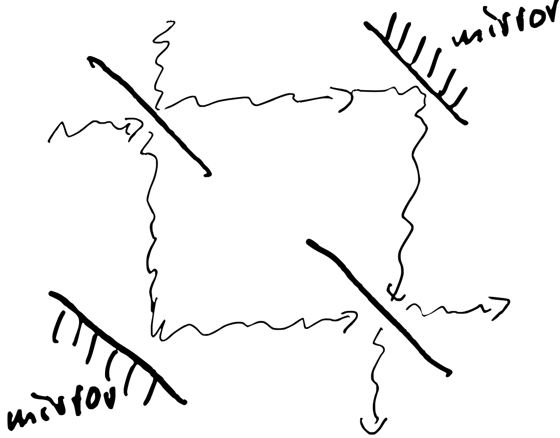
$$B|0\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = |+\rangle$$

$$B|1\rangle = -\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = -|-\rangle$$



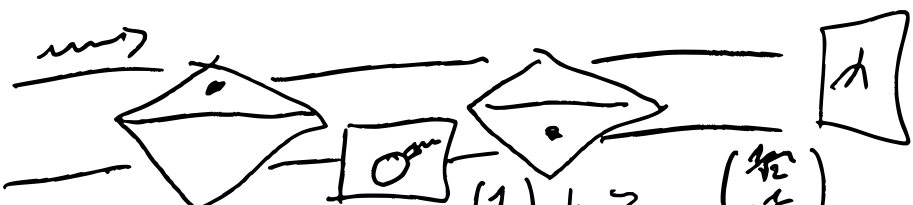
$$B^{+} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

$= B^{+} \cdot B = 1$



$$B = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$

$$B^{+} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}$$



$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad$ $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ $\quad$ $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = |up\rangle$ $\quad$ $\begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$

$Pr[boom] = \frac{1}{2}$

$Pr[not\ boom] = \frac{1}{2}$

$Pr[up] = \frac{1}{2}$

$Pr[down] = \frac{1}{2}$

$|up\rangle$ $\quad$ $\frac{1}{\sqrt{2}}|up\rangle + \frac{1}{\sqrt{2}}|down\rangle$

measurement: up or down?

Outcome	bomb	no bomb
up	1/4	1
down	1/4	0
boom	1/2	0

If we see "boom",
know: "bomb"

If we see "up": don't really know

If we see "down": know: "bomb"

Complete measurements

Described by N unit vectors
$|\phi_1\rangle, \dots, |\phi_N\rangle \in \mathbb{C}^N$
s.t. they are mut. ortho

| $= \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

— $= \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

/ $= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

\ $= \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

$\Pr[\text{outcome } i] = ?$
(given state $|\psi\rangle$)

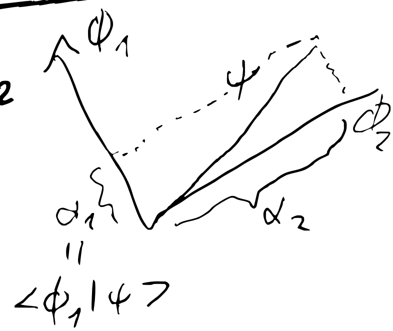
$|\phi\rangle, |\psi\rangle$ orthog.
iff $\langle\phi|\psi\rangle = 0$
$\quad\quad\quad \parallel$
$\sum_i \overline{\phi_i}\,\psi_i$

$|\psi\rangle = \alpha_1 |\phi_1\rangle + \alpha_2 |\phi_2\rangle + \dots$

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \end{pmatrix}$$

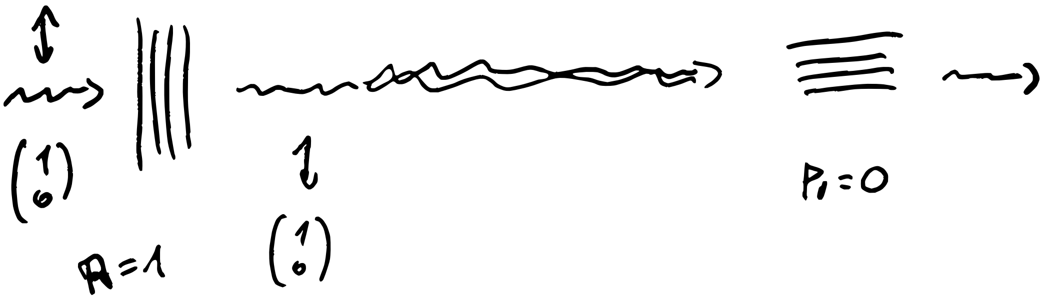
$\Pr[\text{outcome } i] = |\alpha_i|^2$

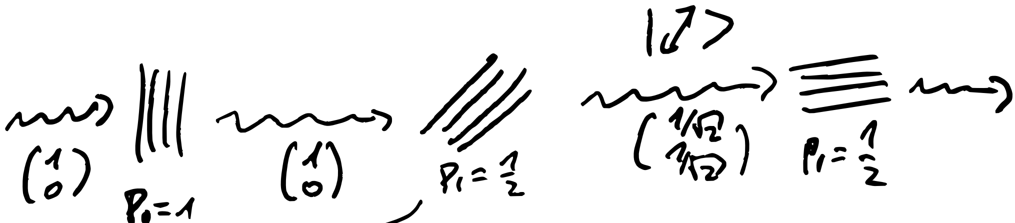
$\Pr[\text{outcome } i] = |\langle\phi_i|\psi\rangle|^2$



Post-meas-state?

$|\phi_i\rangle$





Compl. meas.

$$|\nearrow\rangle = \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$$

$$|\nwarrow\rangle = \begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{pmatrix}$$

$$P_1[\text{outcome } \nearrow]$$

$$= \left|\left\langle \begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix} \right\rangle\right|^2$$

$$= \left|\frac{1}{\sqrt{2}}\cdot 1 + \frac{1}{\sqrt{2}}\cdot 0\right|^2 = \frac{1}{2}$$

Total prob. of photon getting through: $1/4$